A Solution to the Proton Radius
“Puzzle”

Blaine Norum
*University of Virginia*

with

Douglas Higinbotham, Dave Meekins, Brad Sawatzky, Vincent Lin
*Jefferson Lab*

and

Al Amin Kabir
*Kent State University*
The “Puzzle”

Dominated by Atomic Hydrogen

BEFORE model-dependent correction

Proton radius (fm)

- Orsay 1962
- Stanford 1963
- Saskatoon 1974
- Borkowski 1975
- Mainz 1980
- CODATA 2002
- Sick 2003
- CODATA 2006
- CREMA 2010
- Mainz 2010
- Hill 2010
- CODATA 2010
- JLab 2011
- Lorenz 2012
- Sick 2012
- CREMA 2013
- Lorenz 2014

Muonic

??
Elastic Electron Scattering

\[
\frac{d\sigma}{d\Omega} = \left( \frac{d\sigma}{d\Omega} \right)_{\text{Mott}} \frac{E'}{E} \left[ \frac{G_E^2(Q^2) + \tau G_M^2(Q^2)}{1 + \tau} + 2\tau G_M^2(Q^2) \tan^2 \frac{\theta}{2} \right]
\]

where \( G_E(0) \equiv 1 \) and \( G_M(0) \equiv \mu_p \)

Cross section at small \( \theta \) and small \( Q^2 \) dominated by \( G_E \)
Cross section at large \( \theta \) and large \( Q^2 \) dominated by \( G_M \)

In general, \( G_E \) and \( G_M \) can be separated by Rosenbluth or asymmetry measurements.
Elastic Electron Scattering

\[ G_E(Q^2) = 1 + \sum_{n \geq 1} \frac{(-1)^n}{(2n + 1)!} \langle r^{2n} \rangle Q^{2n} = 1 - \frac{1}{6} \langle r^2 \rangle Q^2 + \frac{1}{120} \langle r^4 \rangle Q^4 + \ldots \]

and \( r_p \equiv \sqrt{\langle r^2 \rangle} \)

Notes:

1) the \( Q^2 \) term uniquely defines the radius \( r_p \); connection between statistical analysis and Physics.
2) in the absence of *ab initio* knowledge of the functional form of \( \rho(r) \), each term is functionally separate; that is, it is a particular moment of \( \rho \).
3) if *any* “model” (monopole, dipole, Gaussian, Pade’, …) is used then terms beyond those explicitly fitted are related in a predetermined way; potential conflict with 1).
Saskatchewan ’74 & Mainz ‘80

\[ G_E^{\text{fit}}(Q^2) = n_0 \left[ 1 + a_1 Q^2 + a_2 Q^4 \right] \]

Data:
- SAL74 \([0.15 \leq Q^2 \leq 0.8]\) fm\(^{-2}\)
- MAMI80 \([0.14 \leq Q^2 \leq 1.4]\) fm\(^{-2}\)

\[ \rightarrow r_p = \sqrt{-6a_1} = 0.873(39) \]

Noted that changing highest \(Q^2\) point by \(1\sigma\) changed \(r_p\) by 0.01 = \(\frac{1}{4}\) of discrepancy. Too sensitive?
A standard problem is to quantify when to stop adding terms to a nested statistical model (e.g. a power series). Adding unnecessary terms distorts parameters and increases errors (Phys. Rev. D86). Do we need to add a $j$th term in a fit of $N$ data points?

$$ F = \frac{\chi^2(j-1) - \chi^2(j)}{\chi^2(j)} (N - j - 1) $$

In present case, $j=3$: test yields NO! Only $j = 2$ (linear) function

$\rightarrow r_p = 0.84(1)$
Mainz 2014 Data

Beautiful set of (e,e) data:

6 beam energies –
  180 MeV
  315 MeV
  450 MeV
  485 MeV
  720 MeV
  855 MeV

3 spectrometers
Did not follow a standard statistical method, such as an F-test, to determine number of parameters. Instead, the authors just state $\chi^2_{r} < 1600$ are “analyses with the good models.”

<table>
<thead>
<tr>
<th>Model</th>
<th>$\chi^2$</th>
<th>Number of param.</th>
<th>$\chi^2_{\text{red}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Single dipole</td>
<td>3422</td>
<td>$2 \times 1 + 31$</td>
<td>2.4635 $\times$</td>
</tr>
<tr>
<td>* Double dipole</td>
<td>1786</td>
<td>$2 \times 3 + 31$</td>
<td>1.2893</td>
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<tr>
<td>Polynomial</td>
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<td>$2 \times 10 + 31$</td>
<td>1.1399</td>
</tr>
<tr>
<td>Poly. + std. dipole</td>
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<td>$2 \times 10 + 31$</td>
<td>1.1400</td>
</tr>
<tr>
<td>Poly. $\times$ std. dipole</td>
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<td>$2 \times 8 + 31$</td>
<td>1.1436</td>
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<tr>
<td>Inv. poly.</td>
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<td>$2 \times 7 + 31$</td>
<td>1.1406</td>
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<tr>
<td>Spline</td>
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<td>$2 \times 8 + 31$</td>
<td>1.1385</td>
</tr>
<tr>
<td>Spline $\times$ std. dipole</td>
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<td>$2 \times 7 + 31$</td>
<td>1.1403</td>
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<tr>
<td>Friedrich-Walcher</td>
<td>1598</td>
<td>$2 \times 7 + 31$</td>
<td>1.1588</td>
</tr>
<tr>
<td>* ext. Gari-Krümpelmann</td>
<td>1759</td>
<td>$14 + 31$</td>
<td>1.2777</td>
</tr>
</tbody>
</table>

Note no penalty terms even though data and errors multiplied by the normalizations.

* F test is for nested models only, other test for non-nested models.
Mainz 2014 Fitting Results ($G_E$ & $G_M$)

It is the back angle data (i.e., $G_M$ dominated) that is driving the “need” for higher order fits.

$$
\frac{d\sigma/d\Omega}{(d\sigma/d\Omega)_{\text{Mot}}} = \frac{G_E^2(Q^2) + \tau G_M^2(Q^2)}{1 + \tau} + 2\tau G_M^2(Q^2)\tan^2\frac{\theta}{2} = \frac{\varepsilon G_E^2 + \tau G_M^2}{\varepsilon(1+\tau)} = \frac{G_E^2}{(1+\tau)} + \frac{\tau G_M^2}{\varepsilon(1+\tau)}
$$

where $\varepsilon = \left[1 + 2(1 + \tau)\tan^2\frac{\theta}{2}\right]^{-1}$ and $\tau = \frac{Q^2}{4M_{\rho}^2}$

$G_E = 0.84 \text{ fm}$!

$G_M = 0.80 \text{ fm}$!
Other approaches

Just relying on $\chi^2$ can be misleading and F-test not appropriate for non-nested models

• Analysis of Variance (ANOVA) Tables
  – Checking the significance of each fit parameter

• Akaike Information Criterion (AIC) $[= 2k - 2\ln(L)]$
  – Rigorous way a dealing with the trade-off between the goodness of the fit and the complexity of the model.
  – Unlike the F-test, does not require nested models.

• Stepwise Regression
  – Semi-automated way of building a statistical model by successively adding or removing variables.

• R Programming Language
  – Built for statistical computing and graphics.
  – Example polynomial Stepwise Regression posted on line
\( r_p \) vs. Order of Polynomial Fits

Fits to the full 1422 points of the Mainz 2014 Data Set Using A Python Fitting Code That Reproduces The PRC Results

Using counting statistics only, even with 51 parameters \( \chi^2_r \) never gets to unity.
If the uncertainties are increase by 0.1\%, then 4\textsuperscript{th} order \( \chi^2_r \) already less then unity.

\[
R^2 \equiv 1 - \frac{SS_{\text{res}}}{SS_{\text{tot}}} = 1 - \frac{\sum (y_i - f_i)^2}{\sum (y_i - \bar{y})^2}
\]
Using AIC, one rejects the 6th order polynomial (j=7). F-test gives the same result.
Stepwise Regression of Carl & Keith’s GE Results

\[ G_E(Q^2) = n_0 \left[ 1 + a_1 Q^2 + a_2 Q^4 \right] \]

- \( n_0 = 0.9988(4) \)
- \( a_1 = -0.1173(11) \rightarrow r_p = \sqrt{-6a_1} = 0.839(4) \)
- \( a_2 = +0.0064(21) \)
**Bottom Line**

Bernauer et al. 10\textsuperscript{th} Order Poly
Stepwise Regression
Linear Fit

Stepwise Regression scanned power series to 11\textsuperscript{th} Order
Other analyses

World Cross Section Data

At very Low $Q^2$ $G_e$ dominates the cross section
At very high $Q^2$ $G_m$ dominates the cross section.

Adding Polarization Data

0.6 $fm^{-2} \approx 0.025 GeV^2$

“Proton Radius Puzzle” in 1975 !?


\[
G_{E,M}(q^2) = 1 - \frac{1}{6} \langle r_{E,M}^2 \rangle |q|^2 + \frac{1}{120} \langle r_{E,M}^4 \rangle |q|^4 - + \ldots , \tag{6}
\]

For \( q^2 < 0.9 \text{ fm}^{-2} \) the contributions of the higher terms in the expansion (6) are negligible and the series can be truncated to give \( G_E(q^2) = \delta + \beta q^2 \). From fitting this expression to the form factors of fig. 5, the solid line of fig. 5 has been obtained. The best fit parameters were \( \delta = 0.994 \pm 0.002 \) and \( \beta = -0.118 \pm 0.004 \text{ fm}^2 \). The reduced \( \chi^2 \) was 0.5. The result of the fit did not depend significantly on the fitted \( q^2 \) range. This was checked by fitting additionally the \( G_E \) values of table 2 up to 1.2 fm\(^{-2} \). The addition of a \( q^4 \) term to the fit formula did not improve the fit, moreover the error of the additional parameter turned out to be larger than its value. The best fit value of the parameter \( \delta \) is well within the normalization error of the \( G_E \) values. The best fit value of the parameter \( \beta \) gives a proton r.m.s. radius of \( \langle r_E^2 \rangle^{1/2} = 0.84 \pm 0.02 \text{ fm} \). This value is higher than the dipole value of 0.81 fm, but within the error limits it is compatible with the result (0.81 \( \pm \) 0.04 fm) of a similar experiment carried out at Saskatoon [7]. The radius values reported in (I) resulted from neutron data from the same experiment as the proton data.
Fitting with Standard Functions

Using the “old” Stanford, Jlab, Mainz, Saskatoon data along with the Mainz 2014 “Rosenbluth” $G_E$ Results

Data shown with $1/\sqrt{N}$ errors only.
Gray area is a 0.5% systematic error band.

“… essentially, all models are wrong, but some are useful.” – George E.P. Box
And, the dipole model is AMAZING!
Conclusions

• We applied standard statistical modeling techniques (F-tests, AIC, Stepwise Regression) to determine the function to fit a given set of data.
• We systematically found a value for $r_p$ that was consistent with the value obtained from muonic atom measurements [0.8409(4)] and statistically inconsistent with the CODATA value [0.878(5)].
• Only remaining anomalies are the atomic hydrogen measurements.

Note: R based Stepwise Regression Code Posted Along With Example Data Sets http://jeffersonlab.github.io/model-selection/

References:
Andrae, Schulze-Hartung and Melchior, arXiv:1012.3754. (Do’s & Don’ts of $\chi^2$)
Higinbotham et al., Accepted for publication in Phys. Rev. C., arXiv:1510.01293.