Basis Light-Front Quantization Approach to Heavy Quarkonium

Yang Li

With Lekha Adhikari, Guangyao Chen,
Pieter Maris, James P. Vary, Xingbo Zhao

Department of Physics and Astronomy,
Iowa State University, Ames, IA

International Conference on the Structure of Baryons
Tallahassee, FL, May 16–20, 2016
Dirac’s Forms of Relativistic Dynamics

Front form defines QCD on the light front (LF) \( x^+ \triangleq t + z = 0 \).

\[
P^\pm \triangleq P^0 \pm P^3, \quad \vec{P}^\perp \triangleq (P^1, P^2), \quad x^\pm \triangleq x^0 \pm x^3, \quad \vec{x}^\perp \triangleq (x^1, x^2), \quad E^i = M^+ i, \quad E^+ = M^{+-}, \quad F^i = M^{-i}, \quad K^i = M^{0i}, \quad J^i = \frac{1}{2} \epsilon^{ijk} M^{jk}.
\]

<table>
<thead>
<tr>
<th>instant form</th>
<th>front form</th>
<th>point form</th>
</tr>
</thead>
<tbody>
<tr>
<td>time variable</td>
<td>( t = x^0 )</td>
<td>( x^+ \triangleq x^0 + x^3 )</td>
</tr>
</tbody>
</table>

![Quantization surface](image1)

- **Hamiltonian**: \( H = P^0 \)
- **Kinematical**: \( \vec{P}, \vec{J} \)
- **Dynamical**: \( \vec{K}, P^0 \)
- **Dispersion relation**: \( p^0 = \sqrt{p^2 + m^2} \)

\[
P^- \triangleq P^0 - P^3, \quad \vec{P}^\perp, P^+, \vec{E}^\perp, E^+, J_z
\]

\[
p^- = (p^2_{\perp} + m^2)/p^+ \quad \Rightarrow \quad p^\mu = mv^\mu (v^2 = 1)
\]
Light-Front QCD \((A^+ = 0)\)

hadron spectrum and structure

\[(P^+ \hat{P}^- - \vec{P}_\perp^2) |\psi_h\rangle = M_h^2 |\psi_h\rangle\]

quantum evolution in strong fields

\[i \frac{\partial}{\partial x^+} |\psi(x^+)\rangle = \frac{1}{2} \hat{P}^- |\psi(x^+)\rangle\]

HFQCD vertices (in light-cone gauge \(A^+ = 0\))

- Hamiltonian formalism provides non-perturbative approaches to QCD bound-state problems in Minkowski space. It also allows real-time access to the quantum evolution of fields. \[\text{[Zhao '13]}\]
- LF dynamics resembles nonrelativistic dynamics and the vacuum is simple. "light-cone Hamiltonian" \(H_{LC} \triangleq P^+ \hat{P}^- - \vec{P}_\perp^2\)
- LF wavefunctions are frame independent and provide intrinsic information of the system. LFWF \(\neq\) equal-time WF in rest frame \[\text{[Järvinen '05]}\]
- LF quantization is closely related to physics in infinite momentum frame in Deep Inelastic Scattering. \[\text{[e.g., Burkardt '96]}\]
Basis Light-Front Quantization (BLFQ) [Vary, Phys.Rev.C '10]

Fock space is the most intuitive representation of LFQCD:

\[ |\psi_h\rangle = \psi_{h/q\bar{q}}|q\bar{q}\rangle + \psi_{h/q\bar{q}g}|q\bar{q}g\rangle + \psi_{h/q\bar{q}gg}|q\bar{q}gg\rangle + \cdots \]

- LF Tamm-Dancoff coupled integral equations
  - few-body approach [e.g., Perry '90, Karmanov '08, Li '15]

- Direct diagonalization: large sparse matrix eigenvalue problem
  - many-body approach: DLCQ, BLFQ, ... [e.g., Pauli '89, Vary '10]
  - in parallel with \textit{ab initio} nuclear structure calculations [Barrett '13]
    - configuration interaction, Green function Monte-Carlo, coupled cluster ...
  - need effective eigensolvers suitable for HPC [work in progress!]

- Collective modes [e.g., Vary '05, Misra '00, More '12, Chabysheva '12]
  - coherent basis, LF coupled cluster, ...

BLFQ implements the LFQCD Hamiltonian approach: [Vary '10]

- It adopts basis function expansion and basis regularization.
- It exploits the kinematic symmetries of the Hamiltonian.
- \textbf{Time}-dependent Basis Light-Front Quantization (tBLFQ). [Zhao '13]

Key insight: nonrelativistic and light-front Hamiltonian problems have much in common.
Heavy Quarkonium

Applications of BLFQ in QED:

- electron anomalous magnetic moment  
  [Zhao '13]
- positronium spectrum, form factor and GPDs  
  [Wiecki '15, Adhikari '16]

First application of BLFQ in QCD: heavy quarkonium

- Ideal laboratory to study the interplay between perturbative and non-perturbative QCD:  
  [Brambilla '11]
  - extensive experimental measurements: BaBar, Belle, CLEO, LHC ...
  - many mysteries: XYZ, $P_c^+$, molecules, quark-gluon hybrids ...
  - important for: SM parameters, beyond SM, Dark Matter ...

- Physical picture:  
  [e.g., Eichten '75, Godfrey '83]
  - non-relativistic potential model: confinement plus Coulomb;
  - relativity necessary for getting the hyperfine structure;

- Theoretical approaches:  
  [Brambilla '14]
  Effective Field Theory, Lattice QCD, Dyson-Schwinger/Bethe-Salpeter Equation ...

Yang Li, Iowa State U, May 15, 2016

Baryons 2016, Tallahassee, FL
As mentioned, we need effective eigensolvers suitable for modern high performance computing. Can we find a way to reduce the Hilbert space?

- Fock sector truncation, effective Hamiltonian method etc  

\[ H_{\text{eff}} = \mathcal{P}H_0\mathcal{P} + \mathcal{P}HQ \frac{1}{\frac{1}{2}(\epsilon_i + \epsilon_f) - QH_0Q\mathcal{P}} \]

However, this is only suitable for QCD at short distance (and QED).

- For long-distance physics, we adopt a confining potential inspired by light-front holographic QCD

\[ V(\zeta_\perp) = \kappa^4 \zeta_\perp^2 + \text{const.} \quad (\zeta_\perp = \sqrt{x(1-x)r_\perp}) \]

- AdS/QCD: first approximation to QCD inspired by AdS/CFT
- soft-wall AdS/QCD produces Regge trajectory  

[Karch '06]
- LF holography relates AdS/QCD to LF Schrödinger equation
- successful applications: spectrum, form factors, $\beta$-function, ...

Effective Hamiltonian II


Quark masses and longitudinal dynamics:

► Soft-wall confinement is purely transverse and were derived for massless quarks

► Invariant mass ansatz: \[ \frac{k^2}{x(1-x)} \rightarrow \frac{k^2 + m_q^2}{x} + \frac{k^2 + m_{\bar{q}}^2}{1-x} \]

We proposed a longitudinal confinement:

► It generates distribution amplitudes that match pQCD asymptotics:

\[ \chi_\ell(x) \sim x^\alpha (1-x)^\beta P_\ell^{(\alpha,\beta)} (2x - 1) \]

► In massless limit, it restores the soft-wall model

► In nonrelativistic limit, it sits on equal footing with the transverse confinement

transverse & longitudinal confinements form a 3D HO potential

► No extra free parameters

\[
H_{\text{eff}} = \frac{k^2 + m_q^2}{x} + \frac{k^2 + m_{\bar{q}}^2}{1-x} + \kappa^4 \zeta^2 - \frac{\kappa^4}{(m_q + m_{\bar{q}})^2} \partial_x \left( x(1-x) \partial_x \right) + V_g
\]

\[ \text{dilaton field} \sim e^{-\kappa^2 z^2} \]

new for heavy quarkonium!

Yang Li, Iowa State U, May 15, 2016

Baryons 2016, Tallahassee, FL
The Hamiltonian is analytically solvable without the one-gluon exchange:

- **Transverse:** 2D HO in holographic variables $\phi_{nm}(\vec{k}_\perp/\sqrt{x(1-x)})$

- **Longitudinal:** $\chi_\ell(x) = x^{1/2}\alpha (1-x)\frac{1}{2}\beta P_\ell^{(\alpha, \beta)}(2x-1)$
  
  \[ \alpha = 2m_\bar{q}(m_q + m_\bar{q})/\kappa^2, \quad \beta = 2m_q(m_q + m_\bar{q})/\kappa^2, \quad P_\ell^{(a, b)}(z) \quad \text{Jacobi polynomials} \]

- **Mass eigenvalues:**
  \[ M_{nm\ell}^2 = (m_q + m_\bar{q})^2 + 2\kappa^2(2n + |m| + \ell + 3/2) + \frac{\kappa^4}{(m_q+m_\bar{q})^2} \ell(\ell + 1) \]

We adopt these functions (soft-wall LFWFs) as the basis:

\[ \psi_{h/q\bar{q}}(\vec{k}_\perp, x, s, \bar{s}) = \sum_{n,m,l} \Psi_{h/q\bar{q}}(n, m, l, s, \bar{s}) \phi_{nm}(\vec{k}_\perp/\sqrt{x(1-x)}) \chi_\ell(x) \]

- implement LF holographic QCD for first approximation
- transverse 2D HO functions are scalable in the many-body sector (factorization of c.m. motion) \[\text{[Li '13]}\]
- basis truncation: $2n + |m| + 1 \leq N_{\text{max}}, \ell \leq L_{\text{max}}$
- quantum number identification (esp. mirror parity) \[\text{[Soper '72]}\]

We fix $\alpha_s$ and fit $\kappa, m_q$ to the experimentally measured masses.
Mass Spectroscopy

Masses show weak $m_J$ dependence due to the violation of rotational symmetry. We use boxes to indicate the spread of masses (dashed bars: averaged masses).

<table>
<thead>
<tr>
<th></th>
<th>$\alpha_s$</th>
<th>$\mu_g$ (GeV)</th>
<th>$\kappa$ (GeV)</th>
<th>$m_q$ (GeV)</th>
<th>$\delta \overline{M}$ (MeV)</th>
<th>$N_{\text{max}} = L_{\text{max}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c\bar{c}$</td>
<td>0.3595</td>
<td>0.02</td>
<td>0.938</td>
<td>1.522</td>
<td>52 (8 states)</td>
<td></td>
</tr>
<tr>
<td>$b\bar{b}$</td>
<td>0.2500</td>
<td></td>
<td>1.490</td>
<td>4.763</td>
<td>50 (14 states)</td>
<td>24</td>
</tr>
</tbody>
</table>
Mass Spectroscopy

Masses show weak $m_J$ dependence due to the violation of rotational symmetry. We use boxes to indicate the spread of masses (dashed bars: averaged masses).

<table>
<thead>
<tr>
<th></th>
<th>$\alpha_s$</th>
<th>$\mu_g$ (GeV)</th>
<th>$\kappa$ (GeV)</th>
<th>$m_q$ (GeV)</th>
<th>$\delta \bar{M}$ (MeV)</th>
<th>$N_{max} = L_{max}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c\bar{c}$</td>
<td>0.3595</td>
<td>0.02</td>
<td>0.938</td>
<td>1.522</td>
<td>52 (8 states)</td>
<td>24</td>
</tr>
<tr>
<td>$b\bar{b}$</td>
<td>0.2500</td>
<td>1.490</td>
<td>4.763</td>
<td></td>
<td>50 (14 states)</td>
<td></td>
</tr>
</tbody>
</table>
Mass Spectroscopy: Improvement

Running coupling implements important UV physics:

\[ \alpha_s(Q^2) = \frac{\alpha_s(M_Z^2)}{1 + \alpha_s(M_Z^2)\beta_0 \ln \left( \frac{\mu_{IR}^2 + Q^2}{\mu_{IR}^2 + M_Z^2} \right)}, \]

\[ V_g = -\frac{4}{3} \times \frac{4\pi \alpha_s(Q^2)}{Q^2} \bar{u}_\sigma' \gamma^\mu u_\sigma \bar{v}_\tau' \gamma_\mu v_\tau', \]

The running coupling improves the one-gluon exchange kernel. The \( m_J \)-dependence of masses becomes weaker. The overall mass spectrum is improved: \( \delta \bar{M} = 28 \text{ MeV (charmonium)} \) \( 43 \text{ MeV (bottomonium)} \) \( (N_{\text{max}} = L_{\text{max}} = 16, \alpha_s(0) = 0.6) \)

Yang Li, Iowa State U, May 15, 2016
Model Parameters and Regulator Sensitivity

- For HO basis, $\Omega_{\text{IR}} \sim b/\sqrt{N_{\text{max}}}$, $\Omega_{\text{UV}} \sim b\sqrt{N_{\text{max}}}$. [Coon '12]

- Positronium: continuum limit $N_{\text{max}} \to \infty$, $L_{\text{max}} \to \infty$, $\mu_g \to 0$ can be reached through successive extrapolations. [Wiecki '15, Vary '15]

- Quarkonium, $N_{\text{max}} = L_{\text{max}} = 8, 16, 24$
  - $\kappa, m_q$ refitted and turned out to be very close ($\lesssim 1\%$ changes).
  - The r.m.s. mass deviations are also comparable.

<table>
<thead>
<tr>
<th></th>
<th>$\alpha_s$</th>
<th>$\mu_g$ (GeV)</th>
<th>$\kappa$ (GeV)</th>
<th>$m_q$ (GeV)</th>
<th>$\delta M$ (MeV)</th>
<th>$N_{\text{max}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c\bar{c}$</td>
<td>0.3595</td>
<td>0.02</td>
<td>0.963</td>
<td>1.492</td>
<td>56 (8 states)</td>
<td>8</td>
</tr>
<tr>
<td>$b\bar{b}$</td>
<td>0.2500</td>
<td></td>
<td></td>
<td></td>
<td>55 (14 states)</td>
<td></td>
</tr>
<tr>
<td>$c\bar{c}$</td>
<td>0.3595</td>
<td>0.02</td>
<td>0.950</td>
<td>1.510</td>
<td>52 (8 states)</td>
<td>16</td>
</tr>
<tr>
<td>$b\bar{b}$</td>
<td>0.2500</td>
<td></td>
<td></td>
<td></td>
<td>51 (14 states)</td>
<td></td>
</tr>
<tr>
<td>$c\bar{c}$</td>
<td>0.3595</td>
<td>0.02</td>
<td>0.938</td>
<td>1.522</td>
<td>52 (8 states)</td>
<td>24</td>
</tr>
<tr>
<td>$b\bar{b}$</td>
<td>0.2500</td>
<td></td>
<td></td>
<td></td>
<td>50 (14 states)</td>
<td></td>
</tr>
<tr>
<td>$c\bar{c}$</td>
<td>$\alpha_s(Q)$</td>
<td>0.02</td>
<td>0.979</td>
<td>1.587</td>
<td>28 (8 states)</td>
<td>16 (preliminary)</td>
</tr>
<tr>
<td>$b\bar{b}$</td>
<td></td>
<td></td>
<td>1.451</td>
<td>4.890</td>
<td>43 (14 states)</td>
<td></td>
</tr>
</tbody>
</table>
Light-Front Wavefunctions (LFWFs)

LFWFs provides intrinsic information of the structure of hadrons:

▶ Form factors (electromagnetic, gravitational ...)

\[
A(q^2) = \sum_n \int dD_n \sum_{f=1}^n x_f \psi_n^*(\{\vec{k}'_{i\perp}, x_i, \lambda_i\}_f) \psi_n(\{\vec{k}_{i\perp}, x_i, \lambda_i\}_f)
\]

\[
\vec{k}'_{i\perp} = \begin{cases} 
\vec{k}_{i\perp} + (1 - x_i) \vec{q}_{\perp}, & \text{for struck partons} \\
\vec{k}_{i\perp} - x_i \vec{q}_{\perp}, & \text{for spectators.}
\end{cases}
\]

▶ Distributions (hadron tomography)
Form factors are defined from the matrix elements of the “good current”,
\[ I^+_{\lambda, \lambda'}(Q^2) = \langle P', \lambda' | J^+(0) | P, \lambda \rangle / (2P^+) \]
where \( q = P' - P, \ Q^2 = -q^2 \).

- Impulse approximation with only the two-body contribution.
- GK prescription for (axial-)vectors [Grach '84]
- pQCD asymptotics: \( Q^2 F_P(Q^2) \sim 8\pi \alpha_s f_P^2 \) [Lepage & Brodsky '80]
The charge radius:

\[ \langle r^2 \rangle = -6 \frac{\partial}{\partial Q^2} G_0(Q^2) \bigg|_{Q^2 \to 0}. \]

- test long-distance physics (cf. decay constants)

[DSE: Maris '07; Lattice: Dudeck '06]
Generalized parton distributions (GPDs)

\[ H(x, \zeta, t) = \frac{1}{2} \int \frac{dz^-}{2\pi} e^{ixP^+z^-} \langle P' | \bar{\psi}(-\frac{1}{2} z)\gamma^+ \psi(\frac{1}{2} z) | P \rangle \bigg|_{z^+ = z^\perp = 0} \]

\[ q = P' - P, \quad \zeta = \frac{q^+}{P^+}, \quad t = q^2. \]

DVCS, SIDIS, ..., spin physics

Impact parameter dependent GPDs:

\[ q(x, \vec{b}_\perp) = \int \frac{d^2\Delta_\perp}{(2\pi)^2} e^{i\vec{\Delta}_\perp \cdot \vec{b}_\perp} H(x, \zeta = 0, t = -\Delta_\perp^2). \]

partonic interpretation: \( \int d^2b_\perp \int_0^1 dx |q(x, \vec{b}_\perp)|^2 = 1. \)

Light-front wavefunction representation

\[ 1^1S_0 \quad \text{positronium} \]

\[ 2^1S_0 \]
Decay Constants

\[ \langle 0 | \bar{\psi} \gamma^\mu \gamma^5 \psi | P(p) \rangle = i p^\mu f_P, \]
\[ \langle 0 | \bar{\psi} \gamma^\mu \psi | V_\lambda(p) \rangle = e_\lambda(p) m_V f_V \]

Test “wavefunction at the origin” (cf. charge radius)

Results are in reasonable agreement with experimental measurements as well as Lattice and DSE calculations where available.

Results were extrapolated from \( N_{\text{max}} = L_{\text{max}} = 8, 16, 24, \) and there is some residual regulator dependence.
Diffractive Vector Meson Production

Diffractive VM production in DIS is an important tool for studying the small-\(x\) gluon distribution at a future Electron-Ion Collider.

\[ \gamma^* p \rightarrow J/\psi p \]

In color dipole picture:

\[ A_{\gamma^* p \rightarrow V p} = \int d^2r_\perp \int dz \frac{1}{4\pi} [\psi_V \psi]_{T,L} A_{q\bar{q}} (z, r_\perp, \Delta_\perp) \]

\( (\psi_V \psi)_{T,L} \): overlap of photon/vector meson LFWFs

Initial study: \( ep \) collision with IP-Sat for \( A_{q\bar{q}} \)

\[ \gamma^* p \rightarrow J/\psi p \]

\( \psi_V \): boosted Gaussian, Gaus-LC (\( m_c = 1.4 \text{ GeV} \)) vs. BLFQ (\( m_c = 1.35 \text{ GeV} \))
Diffractive Vector Meson Production

The diffractive VM production tests BLFQ over a dynamical range not covered by the mass spectroscopy and decay constants.

- Provides access to excited states that are well constrained by physical observables (mass spectrum, decay constant etc).
- BLFQ LFWFs could help to discern the advantages and limitations of the dipole models (GBW, IP-Sat, b-CGC etc).
- Beyond Eikonal approximation using tBLFQ with Color Glass Condensate

[Chen, in preparation]
Generalization to Baryons

Basis Light-Front Quantization:

\[
\psi(x^-, \vec{x}^\perp) = \sum_{\alpha} \left[ f_\alpha(x^-, \vec{x}^\perp) b_\alpha + g_\alpha^*(x^-, \vec{x}^\perp) d_\alpha^\dagger \right] \bigg|_{x^+=0}
\]

Steps to implement BLFQ

- Enumerate Fock space basis subject to symmetry constraints and regularizations; (keep the bookkeeping under control: cf. No-Core Shell Model)
  \[
  \sum_i b_i = B, \quad \sum_i e_i = Q, \quad \ldots
  \]
  \[
  \sum_i (m_i + s_i) = J_z
  \]
  \[
  \sum_i k_i = K
  \]
  \[
  \sum_i (2n_i + |m_i| + 1) \leq N_{\text{max}}
  \]
  longitudinal direction
  transverse direction
  global color singlet
  ground-state HO for center-of-mass motion: \( H \to H + \lambda H_{\text{CM}} \)

- Evaluate the LC Hamiltonian operator \( H_{\text{LC}} \) in that basis;
- Diagonalization (Lanczos, QR, ...);
- Evaluate observables using LFWFs;
- Repeat previous steps for new regulators, and extrapolate to continuum limit.

Test cases: electron g-2, positronium

[work in progress]

[Vary '10]

[Honkanen '11, Zhao '14, Wiecki '15]
Generalization to Baryons

The effective interaction can be generalized to the baryon sector:

\[ H_{\text{eff}} = \sum_a \frac{\vec{p}_a^2 + m_a^2}{x_a} - \vec{P}_\perp^2 + \frac{1}{2} \sum_{a,b} V_{ab}^{(2)} + \frac{1}{6} \sum_{a,b,c} V_{abc}^{(3)} + \cdots \]

\[ \text{▶ The soft-wall confinement: } V_{\text{SW}} = \frac{1}{2} \sum_{a,b} x_a x_b (\vec{r}_a \perp - \vec{r}_b \perp)^2. \]

\[ \text{▶ The one-gluon exchange} \]

\[ \text{Jacobi coordinates on the light front (three-body example):} \]

longitudinal: \( x = x_3 , \ \ \chi = \frac{x_2}{1-x_3} \);

transverse momenta: \( \vec{k}_\perp = (1-x_3)\vec{p}_3 \perp - x_3(\vec{p}_1 \perp + \vec{p}_2 \perp) , \ \ \vec{\kappa}_\perp = \frac{x_1\vec{p}_2 \perp - 2\vec{p}_1 \perp}{x_1 + x_2} \);

transverse coordinates: \( \vec{r}_\perp = \vec{r}_3 \perp - \frac{x_1 \vec{r}_1 \perp - x_2 \vec{r}_2 \perp}{x_1 + x_2} , \ \ \vec{\rho}_\perp = \vec{r}_1 \perp - \vec{r}_2 \perp . \)

\[ \text{▶ Taking advantage of the kinematical nature of light-front boosts} \]

\[ V_{\text{SW}} = \kappa^4 x(1-x)\vec{r}_\perp^2 + \kappa^4 (1-x)\chi(1-\chi)\vec{\rho}_\perp^2 \]

\[ \text{▶ The longitudinal confinement} \]

\[ V_L = -\frac{\kappa^4}{(m_1+m_2+m_3)^2} \left[ \partial_x (x(1-x)\partial_x) + \frac{1}{1-x} \partial_\chi (\chi(1-x)\partial_\chi) \right] \]
Conclusion and Outlook

▶ Proposed a model for heavy quarkonium implementing the one-gluon exchange and a confining potential inspired by holographic QCD.
▶ Obtained the mass spectra, evaluated the decay constants, form factors/charge radii, and compared with the available experimental data.
▶ First application of Basis Light-Front Quantization in QCD problems, albeit with generalized holographic confinement.
▶ Results motivate the implementation of the running coupling.
▶ Generalizations to other systems, including the heavy-light, light-light and baryons are imminent.
▶ First step towards \textit{ab initio} hadron structure calculations based on many-fermion dynamics for quarks (MFDq).

Thank you!
Fin

Click here to proceed to backup slides...
Light-Front Dynamics

Distinctive features of light-front dynamics:

- Dispersion relation (cf. non-relativistic dispersion relation)
  \[ p^\mu p_\mu = m^2 \Rightarrow \begin{cases} p^0 = \sqrt{\vec{p}^2 + m^2}, & \text{equal-time} \\ p^- = (\vec{p}_\perp + m^2)/p^+, & \text{light-front} \end{cases} \]

- Spectral condition: \( p^+ \geq 0, \ p^- \geq 0 \) leads to simple vacuum structure [Leutwyler '78]

\[ \propto \delta(p^+_1 + p^+_2 + k^+) \]

- Longitudinal & transverse boost transformations are kinematic

- Suitable for Fock sector expansion [Perry '90]

\[ |\psi_h(P)\rangle = \sum_{n=1}^{\infty} \int dD_n \psi_{h/n}(k_1, \cdots k_n; P) |k_1, k_2, \cdots k_n\rangle \]

LFWFs: \( \psi_{h/n}(k_1, k_2, \cdots k_n; P) \) are boost invariants, only depending on boost-invariant variables: \( x_i \triangleq k_i^+ / P^+, \ \vec{k}_i^\perp \triangleq \vec{k}_i^\perp - x_i \vec{P}_\perp \).
\( x^+ \)-ordered perturbation theory (LF perturbation theory) \([\text{e.g., Brodsky '98}]\)

- all particles (constituents) on their mass-shells
- longitudinal and transverse momenta conserved
- light-front energy NOT conserved ("off the energy shell")
- add energy denominator for intermediate states

\[
\begin{align*}
\frac{1}{s_n - M^2}, \quad s_n &\equiv (k_1 + \cdots + k_n)^2 = \sum_{a=1}^{n} \left( \vec{k}_{a \perp}^2 + m_a^2 \right) x_a, \\
M &\text{ is the mass eigenvalue}
\end{align*}
\]

Extended to non-perturbative regime by introducing the vertex functions
\( \Gamma_n \equiv (s_n - m^2) \psi_n \)
One-Gluon Exchange

- Weak coupling expansion can be applied to short-distance physics
  - justified by asymptotic freedom \( \alpha_s(Q^2) \sim \frac{4\pi}{\beta_0 \ln Q^2 / \Lambda_{\text{QCD}}^2} \)

- One-Gluon exchange interaction

  \[
  H_{\text{eff}} = \mathcal{P} H_0 \mathcal{P} + \mathcal{P} H Q \frac{1}{\frac{1}{2}(\epsilon_i + \epsilon_f) - QH_0Q} QH \mathcal{P}
  \]

- identical to the one-photon exchange in QED except for a color factor \( C_F \)
- obtained from the Bloch method [Okubo '54, Bloch '58, Wilson '74]
- additional unitary transformation can be used to regularize the UV and small-\( x \) singularity in LFQCD Hamiltonian
- one-photon exchange kernel tested in positronium problem [Krautgartner '93, Trittman '97, Lamm '14, Wiecki '15]
One-Photon Exchange

- Artificially large coupling $\alpha = 0.3$
- Adopted transverse 2D Harmonic Oscillator plus longitudinal discretized momentum basis
  - HO basis scale $b$ adjusted to find the variational minimum
  - $m_J = \sum_\alpha (m_\alpha + s_\alpha)$
  - $N_{\text{max}}$ truncation, longitudinal resolution $K$, photon mass $\mu$
- Extensive extrapolation $K \to \infty$, $N_{\text{max}} \to \infty$, $\mu \to 0$.
- Excellent agreement with results obtained by NRQM Schrödinger equation plus $O(\alpha^4)$ perturbative QED correction
Confinement

Semi-Classical Light-Front Schrödinger equation: [Brodsky '05]

\[
\left[ \frac{\vec{k}_\perp^2 + m_q^2}{x(1 - x)} + V(\vec{k}_\perp, x) \right] \psi_{h/q\bar{q}}(\vec{k}_\perp, x) = M_h^2 \psi_{h/q\bar{q}}(\vec{k}_\perp, x)
\]

- Holographic QCD or AdS/QCD [e.g., Erlich '05, Karch '06]
  - inspired by the string/gauge duality or AdS/CFT [Maldacena '98]
  - fields in AdS$_5$ directly matched to hadrons
  - introduce dilaton field $\varphi(z)$ to break the conformal symmetry
    soft-wall model: $\varphi(z) \sim \kappa^2 z^2$ produces the Regge trajectory

- Light-Front Holography relates the semi-classical LF Schrödinger equation to AdS/QCD [Brodsky '06–'15]

  \[\zeta_\perp \triangleq \sqrt{x(1 - x)r^*_\perp} \longleftrightarrow z \text{ (the 5}^{\text{th}} \text{ dimension)}\]

  \[V \longleftrightarrow \frac{1}{2} \varphi''(z) + \frac{1}{4} \varphi'^2(z) - \frac{3}{2z} \varphi'(z)\]

  - the soft-wall confining potential: $V(\zeta_\perp) = \kappa^4 \zeta_\perp^2 + \text{const.}$
  - connection established for arbitrary spin mesons and baryons
  - application: spectrum, form factors, $\beta$-function, ...
Decay Constants with Running Coupling

Decay constants with running coupling (with IR modeling)

\[ \eta_c, \eta'_c, J/\psi, \psi'/\psi(3770)\chi_{c1}, \eta_b, \eta'_b, \eta''_b, \Upsilon, \Upsilon', \Upsilon''(1D), \Upsilon(2D)\chi_{b1}, \chi'_{b1} \]

- Similar quality but the residual regulator dependence is somewhat stronger.
- HO basis is designed for confinement (IR) and is expected to have a slower convergence at UV.
- Need larger \( N_{\text{max}}, L_{\text{max}} \) and a careful study of the UV asymptotics of the LFWFs.
- Renormalization