The Spectrum and Structure of Baryon Excitations from Lattice QCD

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SPECIAL RESEARCH CENTRE FOR THE
SUBATOMIC STRUCTURE OF MATTER

THE UNIVERSITY OF ADELAIDE
Positive Parity Nucleon Spectrum: $\chi$QCD (U. Kentucky) Collaboration
Positive Parity Spectrum: Cyprus (Twisted Mass) Collaboration: Jan. ’14
Outline

Variational Analysis

Understanding and Resolving Discrepancies in the Nucleon Spectrum

Wave Functions of Nucleon Excitations

Isolating the $\Lambda(1405)$ in Lattice QCD

Evidence the $\Lambda(1405)$ is a $\overline{K}N$ molecule

Hamiltonian Effective Field Theory Description of Spectra

The nature of the low-lying $N$ Spectrum

Conclusion
Variational Analysis

- Consider a basis of interpolating fields $\chi_i$
Variational Analysis

- Consider a basis of interpolating fields $\chi_i$
- Construct the correlation matrix

$$G_{ij}(p; t) = \sum_x e^{-ip\cdot x} \text{tr} \left( \Gamma \langle \Omega | \chi_i(x) \chi_j(0) | \Omega \rangle \right) = \sum_\alpha A_\alpha^i A_\alpha^j \exp \left( -E_\alpha(p) t \right).$$
Variational Analysis

• Consider a basis of interpolating fields $\chi_i$

• Construct the correlation matrix

$$G_{ij}(p; t) = \sum_x e^{-i p \cdot x} \text{tr} \left( \Gamma \langle \Omega | \chi_i(x) \bar{\chi}_j(0) | \Omega \rangle \right)$$

$$= \sum_{\alpha} A_i^\alpha A_j^{\dagger \alpha} \exp (-E_\alpha(p) t).$$

• Seek linear combinations of the interpolators \( \{ \chi_i \} \) that isolate individual energy eigenstates, $\alpha$, at momentum $p$:

$$\phi^\alpha = v_i^\alpha(p) \chi_i, \quad \bar{\phi}^\alpha = u_i^\alpha(p) \bar{\chi}_i.$$
Variational Analysis

• When successful, only state $\alpha$ participates in the correlation function, and one can write recurrence relations

$$G(p; t_0 + \delta t) \ u^\alpha(p) = e^{-E_\alpha(p) \delta t} \ G(p; t_0) \ u^\alpha(p)$$

$$v^\alpha T(p) \ G(p; t_0 + \delta t) = e^{-E_\alpha(p) \delta t} \ v^\alpha T(p) \ G(p; t_0)$$

a Generalised Eigenvalue Problem (GEVP).
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$$G(p; t_0 + \delta t) u^\alpha(p) = e^{-E_\alpha(p)\delta t} G(p; t_0) u^\alpha(p)$$

$$v^{\alpha T}(p) G(p; t_0 + \delta t) = e^{-E_\alpha(p)\delta t} v^{\alpha T}(p) G(p; t_0)$$

a Generalised Eigenvalue Problem (GEVP).

- Solve for the left, $v^\alpha(p)$, and right, $u^\alpha(p)$, generalised eigenvectors of $G(p; t_0 + \delta t)$ and $G(p; t_0)$. 
Using these optimal eigenvectors, create eigenstate-projected correlation functions

\[ G^\alpha(p; t) = \sum_x e^{-ip \cdot x} \langle \Omega | \phi^\alpha(x) \bar{\phi}^\alpha(0) | \Omega \rangle , \]

\[ = \sum_x e^{-ip \cdot x} \langle \Omega | v_i^\alpha(p) \chi_i(x) \bar{\chi}_j(0) u_j^\alpha(p) | \Omega \rangle , \]

\[ = v^\alpha_T(p) G(p; t) u^\alpha(p) . \]

\[ G^\alpha(p; t) = A_\alpha \exp(-E_\alpha(p) t) . \]
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\[ = v^{\alpha T}(p) G(p; t) u^{\alpha}(p). \]

\[ G^{\alpha}(p; t) = A_{\alpha} \exp(-E_{\alpha}(p) t). \]

Here \( t \) is different from \( t_0 \) and \( \delta t \) and can become large.
Smeared Source to Point Sink Correlation Functions

M_{eff} (GeV) vs. t

- so04
- so09
- so16
- so25
- so35
- so50
- so70
- so100
- so125
- so200
- so400
- so800
- so1600
Further Information

- “Roper Resonance in 2+1 Flavor QCD,”
  M. S. Mahbub, et al. [CSSM],
  arXiv:1011.5724 [hep-lat],

- “Low-lying Odd-parity States of the Nucleon in Lattice QCD,”
  M. Selim Mahbub, et al. [CSSM],
  Phys. Rev. D Rapid Comm. 87 (2013) 011501,
  arXiv:1209.0240 [hep-lat]

- “Structure and Flow of the Nucleon Eigenstates in Lattice QCD,”
  M. S. Mahbub, et al. [CSSM],
  Phys. Rev. D 87 (2013) 9, 094506
  arXiv:1302.2987 [hep-lat].
CSSM Simulation Details

Based on the PACS-CS (2 + 1)-flavour ensembles, available through the ILDG.


- Lattice size of $32^3 \times 64$ with $\beta = 1.90$. $L \approx 3$ fm.
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- 5 pion masses, ranging from 640 MeV down to 156 MeV.
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- Lattice size of $32^3 \times 64$ with $\beta = 1.90$. $L \approx 3$ fm.
- 5 pion masses, ranging from 640 MeV down to 156 MeV.
- The strange quark $\kappa_s$ is held fixed as the light quark masses vary.
  - Changes in the strange quark contributions are environmental effects.
Positive Parity Nucleon Spectrum: CSSM

![Graph showing the spectrum of positive parity nucleons with various data points and trend lines for P-wave N+π and S-wave N+π+π.]
States Tracked via Orthogonal Eigenvectors
Positive Parity Nucleon Spectrum: CSSM

\[ M \text{ (GeV)} \]

\[ m_\pi^2 \text{ (GeV}^2) \]

- Dashed line: P-wave N+\( \pi \)
- Dotted line: S-wave N+\( \pi + \pi \)
Comparison: Hadron Spectrum Collaboration (HSC)

• “Excited state baryon spectroscopy from lattice QCD,”
  R. G. Edwards, J. J. Dudek, D. G. Richards and S. J. Wallace,
CSSM & HSC Comparison: Positive Parity
CSSM & HSC Comparison: Negative Parity CSSM

![Graph showing data points and error bars in various colors, with axes labeled 'M (GeV)' on the y-axis and 'm_π^2 (GeV^2)' on the x-axis.]

Twisted Mass (this work)  J\(^P = \frac{1}{2}^+\)
Clover (this work)  JLAB  BGR
CSSM  Experiment

\(m_N [\text{GeV}]\)

\(m_{\Delta}^2 [\text{GeV}]\)
Positive Parity Spectrum: Cyprus (Twisted Mass) Collaboration: Jan. '14

The graph plots the mass ($m$) versus the squared mass ($m^2$) for different models and experimental data, with $J^P = \frac{1}{2}^+$.

- Twisted Mass (this work)
- Clover (this work)
- CSSM
- JLAB
- BGR
- Experiment

The graph shows a range of data points for various values of $m^2$ and $m$, indicating a comparison of theoretical models with experimental results.
$d$-quark probability density in ground state proton: $m_\pi = 156$ MeV (CSSM)
$d$-quark probability density in first excited proton: $m_{\pi} = 156$ MeV (CSSM)
Positive Parity Nucleon Spectrum: only small smearing: Cyprus

\[ J^P = \frac{1}{2}^+ \]

\[ m^2_{NN} \text{ [GeV]} \]

\[ m_N \text{ [GeV]} \]

Twisted Mass (this work)
Clover (this work)
CSSM
JLAB
BGR
Experiment
Positive Parity Nucleon Spectrum: $r_{RMS}$ smearing of 8.6 lu: Cyprus

\[ J^p = \frac{1}{2}^+ \]
Athens Model Independent Analysis Scheme (AMIAS)

- “Novel analysis method for excited states in lattice QCD: The nucleon case,”
  C. Alexandrou, T. Leontiou, C. N. Papanicolas and E. Stiliaris,
  Phys. Rev. D 91 (2015) 1, 014506
  arXiv:1411.6765 [hep-lat].
Athens Model Independent Analysis Scheme (AMIAS)

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- The Correlation matrix has the spectral decomposition

\[
G_{ij}(t) = \sum_{\alpha=0}^{N_{\text{states}}} A_i^\alpha A_j^{\dagger\alpha} e^{-E_\alpha t} \cdot \quad i, j = 1, \ldots, N_{\text{interpolators}}.
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• Importance sampling is used to select fit parameters, \( A^\alpha_i \) and \( E_\alpha \), with the probability \( \exp(-\chi^2/2) \).
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  - A parallel tempering algorithm is used to avoid local minima traps.
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  - A parallel tempering algorithm is used to avoid local minima traps.
- Parameters are determined by fitting a Gaussian to their probability distributions.
- Increase \( N_{\text{states}} \) until there is no sensitivity to additional exponentials.
Determining $N_{\text{states}} \equiv n_{\text{max}}$ (Cyprus)

- $n_{\text{max}} = 2$
- $n_{\text{max}} = 3$
- $n_{\text{max}} = 4$
- $n_{\text{max}} = 5$

![Graph showing probability distribution for different $n_{\text{max}}$ values.](image)
Analysis of Correlation Matrix is Essential
Sequential Empirical Bayesian (SEB) Analysis: $\chi$QCD Collaboration

$J^P = \frac{1}{2}^+$

![Graph showing data points and fits for different models: Experiment, BGR, CSSM, JLAB Twisted Mass, Clover, $\chi$QCD. The x-axis represents $m^2_{\pi}$ (GeV$^2$) ranging from 0.00 to 0.40, and the y-axis represents $m_N$ (GeV) ranging from 0.00 to 3.5.](image-url)
\( \chi \)QCD & HSC Systematic Comparison - Same Correlators Examined

Note: \( 28 \times 28 = 784 \) correlators versus 1.


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Lowest-lying positive-parity $N^*$ Spectrum
d-quark probability density in ground state proton (CSSM)
$d$-quark probability density in 1st excited state of proton (CSSM)
$d$-quark probability density in $N = 3$ excited state of proton (CSSM)
Comparison with the Simple Quark Model - CSSM
$d$-quark probability density in 1st excited state of proton (CSSM)
d-quark probability density in 1st excited state of proton (CSSM)
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$d$-quark probability density in 1st excited state of proton (CSSM)
$d$-quark probability density in 4th excited state of proton (CSSM)
The Λ(1405) and Lattice QCD

Our 2012 work successfully isolated three low-lying odd-parity spin-1/2 states.


- An extrapolation of the trend of the lowest state reproduces the mass of the Λ(1405).
- Subsequent studies have confirmed these results.

\( \Lambda(1405) \) and Baryon Octet dominated states
Operators Used in $\Lambda(1405)$ Analysis

We consider local three-quark operators with the correct quantum numbers for the $\Lambda$ channel, including

- Flavour-octet operators

$$\chi^8_1 = \frac{1}{\sqrt{6}} \varepsilon^{abc} \left(2(u^a C \gamma_5 d^b)s^c + (u^a C \gamma_5 s^b)d^c - (d^a C \gamma_5 s^b)u^c \right)$$

$$\chi^8_2 = \frac{1}{\sqrt{6}} \varepsilon^{abc} \left(2(u^a C d^b)\gamma_5 s^c + (u^a C s^b)\gamma_5 d^c - (d^a C s^b)\gamma_5 u^c \right)$$

- Flavour-singlet operator

$$\chi^1 = 2\varepsilon^{abc} \left((u^a C \gamma_5 d^b)s^c - (u^a C \gamma_5 s^b)d^c + (d^a C \gamma_5 s^b)u^c \right)$$

- Consideration of 16 and 100 sweeps of gauge-invariant Gaussian smearing provides a $6 \times 6$ correlation matrix.
Flavour structure of the $\Lambda(1405)$

![Graph showing unit eigenvector components for different $m_\pi^2$ values with markers for $16$ sweeps and $100$ sweeps.](#)
The importance of eigenstate isolation (red)
Probing with the electromagnetic current

\[
\ln(G)
\]

Euclidean Time
Only the projected correlator has acceptable $\chi^2$/dof

\[ \chi^2$/dof = 0.55 \]
Strange Magnetic Form Factor of the $\Lambda(1405)$

- Provides direct insight into the possible dominance of a molecular $\bar{K}N$ bound state.
Strange Magnetic Form Factor of the Λ(1405)

• Provides direct insight into the possible dominance of a molecular $\overline{K}N$ bound state.

• In forming such a molecular state, the $\Lambda(u, d, s)$ valence quark configuration is complemented by
  ◦ A $u, \bar{u}$ pair making a $K^- (s, \bar{u})$ - proton $(u, u, d)$ bound state, or
  ◦ A $d, \bar{d}$ pair making a $\overline{K}^0 (s, \bar{d})$ - neutron $(d, d, u)$ bound state.
Strange Magnetic Form Factor of the $\Lambda(1405)$

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- In both cases the strange quark is confined within a spin-0 kaon and has no preferred spin orientation.
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- To conserve parity, the kaon has zero orbital angular momentum.
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  - A $d, \bar{d}$ pair making a $\bar{K}^0 (s, \bar{d})$ - neutron $(d, d, u)$ bound state.
- In both cases the strange quark is confined within a spin-0 kaon and has no preferred spin orientation.
- To conserve parity, the kaon has zero orbital angular momentum.
- Thus, the strange quark does not contribute to the magnetic form factor of the $\Lambda(1405)$ when it is dominated by a $\bar{K}N$ molecule.
$G_M$ for the $\Lambda(1405)$ at $Q^2 \sim 0.16 \text{ GeV}^2$
$G_M$ for the $\Lambda(1405)$ at $Q^2 \sim 0.16\text{ GeV}^2$

$m_\pi = 0.156\text{ GeV}/c^2$

$m_\pi = 0.411\text{ GeV}/c^2$
Hamiltonian Effective Field Theory


Hamiltonian Effective Field Theory Model

- Consider the \( \Lambda(1405) \).
Hamiltonian Effective Field Theory Model

- Consider the Λ(1405).
- The four octet meson-baryon interaction channels of the Λ(1405) are considered: \( \pi \Sigma, \overline{K}N, K\Xi \) and \( \eta \Lambda \).
Hamiltonian Effective Field Theory Model

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- The four octet meson-baryon interaction channels of the $\Lambda(1405)$ are considered: $\pi \Sigma$, $\bar{K}N$, $K\Xi$ and $\eta\Lambda$.
- A single-particle state with bare mass, $m_0 + \alpha_0 m_{\pi}^2$ is also included.
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- In a finite periodic volume, momentum is quantised to $n (2\pi/L)$. 
Consider the $\Lambda(1405)$.  

The four octet meson-baryon interaction channels of the $\Lambda(1405)$ are considered: $\pi \Sigma$, $K N$, $K \Xi$ and $\eta \Lambda$.  

A single-particle state with bare mass, $m_0 + \alpha_0 m^2_\pi$ is also included.  

In a finite periodic volume, momentum is quantised to $n \left(2\pi/L\right)$.  

Working on a cubic volume of extent $L$ on each side, it is convenient to define the momentum magnitudes

$$k_n = \sqrt{n_x^2 + n_y^2 + n_z^2 \frac{2\pi}{L}},$$

with $n_i = 0, 1, 2, \ldots$ and integer $n = n_x^2 + n_y^2 + n_z^2$.  

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Hamiltonian model, $H_0$

Denoting each meson-baryon energy by $\omega_{MB}(k_n) = \omega_M(k_n) + \omega_B(k_n)$, with $\omega_A(k_n) \equiv \sqrt{k_n^2 + m_A^2}$, the non-interacting Hamiltonian takes the form

$$H_0 = \begin{pmatrix} m_0 + \alpha_0 m^2_{\pi} & 0 & 0 & \cdots \\ \omega_{\pi \Sigma}(k_0) & 0 & \omega_{\eta \Lambda}(k_0) & \cdots \\ 0 & \ddots & 0 & \ddots \\ \omega_{\pi \Sigma}(k_1) & \omega_{\eta \Lambda}(k_1) & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$
Hamiltonian model, $H_I$

- Interaction entries describe the coupling of the single-particle state to the two-particle meson-baryon states.
Hamiltonian model, $H_I$

- Interaction entries describe the coupling of the single-particle state to the two-particle meson-baryon states.
- Each entry represents the $S$-wave interaction energy of the $\Lambda(1405)$ with one of the four channels at a certain value for $k_\eta$.

\[
H_I = \begin{pmatrix}
0 & g_{\pi\Sigma}(k_0) & \cdots & g_{\eta\Lambda}(k_0) & g_{\pi\Sigma}(k_1) & \cdots & g_{\eta\Lambda}(k_1) \\
g_{\pi\Sigma}(k_0) & 0 & \cdots & 0 & g_{\eta\Lambda}(k_0) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
g_{\eta\Lambda}(k_0) & 0 & \cdots & 0 & g_{\eta\Lambda}(k_0) & \cdots & 0 \\
g_{\pi\Sigma}(k_1) & g_{\eta\Lambda}(k_1) & \cdots & 0 & g_{\eta\Lambda}(k_1) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
g_{\eta\Lambda}(k_1) & 0 & \cdots & 0 & g_{\eta\Lambda}(k_1) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\end{pmatrix}
\]
Eigenvalue Equation Form

- The eigenvalue equation corresponding to our Hamiltonian model is

\[ \lambda = m_0 + \alpha_0 \, m_\pi^2 - \sum_{M, B} \sum_{n=0}^{\infty} \frac{g_{MB}^2(k_n)}{\omega_{MB}(k_n) - \lambda}. \]

with \( \lambda \) denoting the energy eigenvalue.
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As \( \lambda \) is finite, the pole in the denominator of the right-hand side is never accessed.

The bare mass and the free meson-baryon energies encounter self-energy corrections that lead to avoided level-crossings in the finite-volume energy eigenstates.
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Reference to chiral effective field theory provides the form of \( g_{MB}(k_n) \).
Flavour-singlet couplings between the bare state and the meson-baryon states are constrained by \( SU(3) \)-flavour symmetry and the width of the \( \Lambda(1405) \) resonance.
Hamiltonian model solution and fit

- Flavour-singlet couplings between the bare state and the meson-baryon states are constrained by $SU(3)$-flavour symmetry and the width of the $\Lambda(1405)$ resonance.
- The eigenvalues and eigenvectors of $H$ are obtained via the LAPACK software library routine dgeev.
Flavour-singlet couplings between the bare state and the meson-baryon states are constrained by $SU(3)$-flavour symmetry and the width of the $\Lambda(1405)$ resonance.

The eigenvalues and eigenvectors of $H$ are obtained via the LAPACK software library routine dgeev.

The bare mass parameters $m_0$ and $\alpha_0$ are determined by a fit to the lattice QCD results.
Hamiltonian model fit

\[ E (\text{GeV}) \]

\[ m_{\pi}^2 (\text{GeV}^2) \]

- **Matrix Hamiltonian model**
- Non-int. \( \pi\Sigma \) energy
- Non-int. \( KN \) energy
- Non-int. \( K\Xi \) energy
- Non-int. \( \eta\Lambda \) energy
- \( \Lambda(1405) \) Lattice results
Avoided Level Crossing
Energy eigenstate, $|E\rangle$, basis $|\text{state}\rangle$ composition

$|\langle\text{state}|E\rangle|^2$

$\bar{K}N$

$\pi\Sigma$

$m_0$

156

296

411

570

702

$m_\pi$ (MeV)
Hamiltonian model, $H_I$

- Our approach included a bare state dressed by flavour-singlet coupled meson-baryon channels

$$H_I = \begin{pmatrix}
0 & g_{\pi\Sigma}(k_0) & \cdots & g_{\eta\Lambda}(k_0) & g_{\pi\Sigma}(k_1) & \cdots & g_{\eta\Lambda}(k_1) & \cdots \\
 g_{\pi\Sigma}(k_0) & 0 & \cdots & \vdots & \vdots & 0 & \vdots & \vdots \\
 \vdots & \vdots & 0 & \vdots & \vdots & \vdots & \vdots & \vdots \\
 g_{\eta\Lambda}(k_0) & g_{\eta\Lambda}(k_1) & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 g_{\pi\Sigma}(k_1) & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 g_{\eta\Lambda}(k_1) & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}.$$
Hamiltonian model, $H_I$

- Our approach included a bare state dressed by flavour-singlet coupled meson-baryon channels

\[
H_I = \begin{pmatrix}
0 & g_{\pi \Sigma}(k_0) & \cdots & g_{\eta \Lambda}(k_0) & g_{\pi \Sigma}(k_1) & \cdots & g_{\eta \Lambda}(k_1) & \cdots \\
g_{\pi \Sigma}(k_0) & 0 & \cdots & & & & & \\
\vdots & \vdots & 0 & & & & & \\
0 & & & & & & & \\
g_{\eta \Lambda}(k_0) & g_{\pi \Sigma}(k_1) & & & & & & \\
\vdots & \vdots & \ddots & & & & & \\
g_{\eta \Lambda}(k_1) & & & & & & & \\
\vdots & \vdots & & & & & & \\
\vdots & \vdots & & & & & & \\
\end{pmatrix}
\]

- Ironically, most analyses omit these interactions and instead include only the direct meson-baryon to meson-baryon interactions
  - Weinberg-Tomozawa terms
The two-pole description of the $\Lambda(1405)$

Direct two-to-two particle interactions

- We use the potential derived from Weinberg-Tomozawa term

\[ V_{\alpha,\beta}(k, k') = \frac{g_{\alpha,\beta}^I}{8\pi^2 f^2_\pi} \frac{\omega_{\alpha M}(k) + \omega_{\beta M}(k')}{\sqrt{2\omega_{\alpha M}(k)} \sqrt{2\omega_{\beta M}(k')}} u(k) u(k'). \]
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- Dipole regulator functions \( u(k) \) have a fixed scale of \( \Lambda = 1 \) GeV.
Direct two-to-two particle interactions

• We use the potential derived from Weinberg-Tomozawa term

\[ V_{\alpha,\beta}^{I}(k, k') = \frac{g_{\alpha,\beta}^{I}}{8\pi^{2}f_{\pi}^{2}} \frac{\omega_{\alpha M}(k) + \omega_{\beta M}(k')}{\sqrt{2\omega_{\alpha M}(k)} \sqrt{2\omega_{\beta M}(k')}} u(k) u(k') . \]

• Dipole regulator functions \( u(k) \) have a fixed scale of \( \Lambda = 1 \text{ GeV} \).

• Couplings vanishing in the \( SU(3) \)-flavour symmetry limit are not considered.

• Seven non-trivial couplings are constrained by experimental data in infinite volume.

\[ g_{\pi \Sigma, \pi \Sigma}^{0}, g_{\bar{K}N, \bar{K}N}^{0}, g_{\bar{K}N, \pi \Sigma}^{0}, g_{\pi \Sigma, \pi \Sigma}^{1}, g_{\bar{K}N, \bar{K}N}^{1}, g_{\bar{K}N, \pi \Sigma}^{1}, g_{\bar{K}N, \pi \Lambda}^{1} \]
Direct two-to-two particle interactions

- We use the potential derived from Weinberg-Tomozawa term

\[
V^I_{\alpha,\beta}(k, k') = \frac{g^I_{\alpha,\beta}}{8\pi^2 f^2_\pi} \frac{\omega_{\alpha M}(k) + \omega_{\beta M}(k')}{{\sqrt{2}}\omega_{\alpha M}(k) \sqrt{2}\omega_{\beta M}(k')} u(k) u(k').
\]

- Dipole regulator functions \( u(k) \) have a fixed scale of \( \Lambda = 1 \text{ GeV} \).
- Couplings vanishing in the \( SU(3) \)-flavour symmetry limit are not considered.
- Seven non-trivial couplings are constrained by experimental data in infinite volume.

\[
g^0_{\pi\Sigma,\pi\Sigma} \quad g^0_{\bar{K}N,\bar{K}N} \quad g^0_{\bar{K}N,\pi\Sigma} \quad g^1_{\pi\Sigma,\pi\Sigma} \quad g^1_{\bar{K}N,\bar{K}N} \quad g^1_{\bar{K}N,\pi\Sigma} \quad g^1_{\bar{K}N,\pi\Lambda}
\]

- Finite volume spectrum is then a prediction.
Couplings Constrained by Experiment

(a) $K^- p \rightarrow K^- p$
(b) $K^- p \rightarrow \bar{K}^0 n$
(c) $K^- p \rightarrow \pi^- \Sigma^+$
(d) $K^- p \rightarrow \pi^0 \Sigma^0$
(e) $K^- p \rightarrow \pi^+ \Sigma^-$
(f) $K^- p \rightarrow \pi^0 \Lambda$
Finite Volume $\Lambda$ Spectrum for $L = 3$ fm

\begin{center}
\includegraphics[width=\textwidth]{graph.png}
\end{center}

- non-int. $\pi$-$\Sigma$ energy
- non-int. $\bar{K}$-$N$ energy
- matrix Hamiltonian model
Comparison with $\chi$PT

Direct two-to-two particle interactions & bare state

- Dipole regulator functions $u(k)$ have a fixed scale of $\Lambda = 1$ GeV.
- Seven non-trivial two-to-two particle couplings are considered:

$$g_{\pi \Sigma, \pi \Sigma}^0, g_{\bar{K}N, \bar{K}N}^0, g_{\bar{K}N, \pi \Sigma}^0, g_{\pi \Sigma, \pi \Sigma}^1, g_{\bar{K}N, \bar{K}N}^1, g_{\bar{K}N, \pi \Sigma}^1, g_{\bar{K}N, \pi \Lambda}^1$$
Direct two-to-two particle interactions & bare state

- Dipole regulator functions $u(k)$ have a fixed scale of $\Lambda = 1 \text{ GeV}$.
- Seven non-trivial two-to-two particle couplings are considered:
  \[
  g^{0}_{\pi\Sigma,\pi\Sigma} \quad g^{0}_{\bar{K}N,\bar{K}N} \quad g^{0}_{\bar{K}N,\pi\Sigma} \quad g^{1}_{\pi\Sigma,\pi\Sigma} \quad g^{1}_{\bar{K}N,\bar{K}N} \quad g^{1}_{\bar{K}N,\pi\Sigma} \quad g^{1}_{\bar{K}N,\pi\Lambda}
  \]
- Three new parameters describing bare – two-particle interactions are introduced:
  \[
  m_{B}^{0} \quad g^{0}_{\pi\Sigma,B_{0}} \quad g^{0}_{\bar{K}N,B_{0}}
  \]
Direct two-to-two particle interactions & bare state

- Dipole regulator functions $u(k)$ have a fixed scale of $\Lambda = 1$ GeV.
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- Three new parameters describing bare – two-particle interactions are introduced:

$$m_B^0, g_{\pi \Sigma, B_0}^0, g_{\bar{K}N, B_0}^0$$

- These 10 parameters are constrained by experimental data.
Direct two-to-two particle interactions & bare state

- Dipole regulator functions $u(k)$ have a fixed scale of $\Lambda = 1 \text{ GeV}$.
- Seven non-trivial two-to-two particle couplings are considered

\[
g^0_{\pi \Sigma, \pi \Sigma} \quad g^0_{KN, \bar{K}N} \quad g^0_{KN, \pi \Sigma} \quad g^1_{\pi \Sigma, \pi \Sigma} \quad g^1_{KN, \bar{K}N} \quad g^1_{KN, \pi \Sigma} \quad g^1_{KN, \pi \Lambda}
\]

- Three new parameters describing bare – two-particle interactions are introduced

\[
m^0_B \quad g^0_{\pi \Sigma, B_0} \quad g^0_{KN, B_0}
\]

- These 10 parameters are constrained by experimental data.
- A linear quark mass dependence for the bare mass is constrained by the lattice results.
Couplings and $m_B^0$ Constrained by Experiment

(a) $K^- p \rightarrow K^- p$

(b) $K^- p \rightarrow \bar{K}^0 n$

(c) $K^- p \rightarrow \pi^- \Sigma^+$

(d) $K^- p \rightarrow \pi^0 \Sigma^0$

(e) $K^- p \rightarrow \pi^+ \Sigma^-$

(f) $K^- p \rightarrow \pi^0 \Lambda$
<table>
<thead>
<tr>
<th>Parameter</th>
<th>$SU(3)$</th>
<th>No Bare State</th>
<th>With Bare State</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g^0_{\pi \Sigma, \pi \Sigma}$</td>
<td>-4</td>
<td>-1.53</td>
<td>-1.11</td>
</tr>
<tr>
<td>$g^0_{\bar{K}N, \bar{K}N}$</td>
<td>-3</td>
<td>-2.17</td>
<td>-1.74</td>
</tr>
<tr>
<td>$g^0_{\bar{K}N, \pi \Sigma}$</td>
<td>$\sqrt{3}/2$</td>
<td>0.81</td>
<td>1.26</td>
</tr>
<tr>
<td>$g^0_{\pi \Sigma, B_0}$</td>
<td>-</td>
<td>-</td>
<td>0.24</td>
</tr>
<tr>
<td>$g^0_{\bar{K}N, B_0}$</td>
<td>-</td>
<td>-</td>
<td>0.18</td>
</tr>
<tr>
<td>$m^0_B$/MeV</td>
<td>-</td>
<td>-</td>
<td>1700</td>
</tr>
<tr>
<td>$g^1_{\pi \Sigma, \pi \Sigma}$</td>
<td>-2</td>
<td>-0.14</td>
<td>-0.14</td>
</tr>
<tr>
<td>$g^1_{\bar{K}N, \bar{K}N}$</td>
<td>-1</td>
<td>-0.06</td>
<td>-0.06</td>
</tr>
<tr>
<td>$g^1_{\bar{K}N, \pi \Sigma}$</td>
<td>1</td>
<td>1.36</td>
<td>1.36</td>
</tr>
<tr>
<td>$g^1_{\bar{K}N, \pi \Lambda}$</td>
<td>$\sqrt{3}/2$</td>
<td>0.96</td>
<td>0.96</td>
</tr>
<tr>
<td>$\chi^2$/dof</td>
<td>U$\chi$PT</td>
<td>2.0</td>
<td>2.0</td>
</tr>
<tr>
<td>Pole (MeV)</td>
<td>1379 $- i$ 71</td>
<td>1351 $- i$ 125</td>
<td>1371 $- i$ 101</td>
</tr>
<tr>
<td>Pole (MeV)</td>
<td>1412 $- i$ 20</td>
<td>1422 $- i$ 21</td>
<td>1431 $- i$ 25</td>
</tr>
</tbody>
</table>
Finite Volume \( \Lambda \) Spectrum for \( L = 3 \text{ fm} \)
Finite Volume $\Lambda$ Spectrum for $L = 3$ fm
Finite Volume Dependence of the $\Lambda$ Spectrum

![Graph showing the finite volume dependence of the $\Lambda$ spectrum with various lines representing different energy levels and labels for non-int. $\pi$-$\Sigma$ energy, non-int. $\bar{K}$-$N$ energy, and matrix Hamiltonian model.](image-url)
(a) State 1

(b) State 2

(c) State 3
Low-lying negative-parity $N^*$ Spectrum
Constrain model parameters to experimental data

- Consider $\pi N$ and $\eta N$ and bare state interactions.
- Fit to phase shift and inelasticity.
- Fit yields a pole at $1531 \pm 29 - i 88 \pm 2$ MeV.
- Compare PDG estimate $1510 \pm 20 - i 85 \pm 40$ MeV.
Hamiltonian Model $N^*$ Spectrum: 2 fm
Hamiltonian Model $N^*$ Spectrum: 3 fm
(a) State 1

(b) State 2

(c) State 4
What about the Roper? Lattice results at $L \simeq 3$ fm
Constrain Roper model parameters to experiment

- Consider $\pi N$, $\pi \Delta$ and $\sigma N$ channels, dressing a bare state.
- Fit to phase shift and inelasticity

- Fit yields a pole at $1357 - i 36$ MeV.
- Compare PDG estimate $1365 \pm 15 - i 95 \pm 15$ MeV.
Hamiltonian Model $N'$ Spectrum
(a) State 1  
(b) State 6  
(c) State 7
Conclusions

• A survey of the current literature resolves discrepancies among groups exploring the low-lying nucleon spectrum.
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  ○ The dominance of the $\bar{K}N$ component in finite-volume EFT.
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  ○ Node structure and density is similar to model expectations.
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• Roper of the Constituent Quark Model has been seen on the lattice.
  ○ Node structure and density is similar to model expectations.
• Like the $\Lambda(1405)$, the Roper resonance is dominated by meson-baryon degrees of freedom with little contribution from a quark-model like core.
Artistic view of $\Lambda(1405)$ Structure

or the neutral Roper upon $s \rightarrow d$. 
Supplementary Information

The following slides provide additional information which may be of interest.
Defining the Effective Mass

- At zero momentum, the projected correlator is

\[ G^\alpha(0; t) = A_\alpha \exp(-M_\alpha t) . \]
Defining the Effective Mass

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- Taking the log

\[ \ln G^\alpha(0; t) = \ln(A_\alpha) - M_\alpha t . \]
Defining the Effective Mass

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  \[ G^\alpha(0; \ t) = A_\alpha \ \exp(-M_\alpha \ t). \]

- Taking the log
  \[ \ln G^\alpha(0; \ t) = \ln(A_\alpha) - M_\alpha \ t. \]

- The effective mass is defined as
  \[ M^\alpha_{\text{eff}}(t) = \frac{1}{\Delta t} \ \ln \left( \frac{G^\alpha(t)}{G^\alpha(t + \Delta t)} \right). \]
Defining the Effective Mass

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\[ M^{\alpha}_{\text{eff}}(t) = \frac{1}{\Delta t} \ln \left( \frac{G^\alpha(t)}{G^\alpha(t + \Delta t)} \right). \]

• \( \Delta t = 1 \) or 2 is common.
Positive Parity Nucleon - First Excited State - $m_\pi$: 296 MeV
Positive Parity Nucleon - First Excited State - \( m_\pi: 296 \text{ MeV} \) - \( \chi^2_{\text{dof}}: 0.67 \)
Negative Parity Nucleon - 2nd Excited State - $m_\pi$: 156 MeV

![Graph showing log($G(t)$) versus $t/a$.]
Negative Parity Nucleon - 2nd Excited State - $m_\pi$: 156 MeV - $\chi^2_{dof}$: 0.88
Properties of the Positive Parity Nucleon Spectrum

![Graph showing properties of positive parity nucleon spectrum with various vectors and data points.](image-url)
AMIAS applied to positive-parity Cyprus results

$C^{(1,2,4)}_{11}$

$E_{\text{eff}} \, (\text{GeV})$

$t$

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Dispersion Relation Test for the $\Lambda(1405)$

\[ E = \sqrt{m^2 + p^2} \]

Graph showing points for $E(p)$ and $\sqrt{m^2 + p^2}$ vs. $m_{\pi}^2$ with error bars.
$G_E$ for the $\Lambda(1405)$

When compared to the ground state, the results for $G_E$ are consistent with the development of a non-trivial $\bar{K}N$ component at light quark masses.
$G_E$ for the $\Lambda(1405)$

When compared to the ground state, the results for $G_E$ are consistent with the development of a non-trivial $\bar{K}N$ component at light quark masses.

- Noting that the centre of mass of the $\bar{K}(s, \ell) N(\ell, u, d)$ is nearer the heavier $N$,
  - The anti–light-quark contribution, $\bar{\ell}$, is distributed further out by the $\bar{K}$ and leaves an enhanced light-quark form factor.
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  - The anti–light-quark contribution, $\bar{\ell}$, is distributed further out by the $\bar{K}$ and leaves an enhanced light-quark form factor.
  - The strange quark may be distributed further out by the $\bar{K}$ and thus have a smaller form factor.

$G_E$ for the $\Lambda(1405)$
$G_E$ for the $\Lambda(1405)$
Excited State Form Factors

- The eigenstate-projected three-point correlation function is

\[
G^\mu_{\alpha}(\mathbf{p}', \mathbf{p}; t_2, t_1) = \sum_{x_1, x_2} e^{-i \mathbf{p}' \cdot x_2} e^{i (\mathbf{p}' - \mathbf{p}) \cdot x_1} \times
\]

\[
\times \langle \Omega | v^\alpha_i (\mathbf{p}') \chi_i(x_2) j^\mu(x_1) \chi_j(0) u^\alpha_i(\mathbf{p}) | \Omega \rangle = v^{\alpha T}(\mathbf{p}') G^\mu_{ij}(\mathbf{p}', \mathbf{p}; t_2, t_1) u^\alpha(\mathbf{p})
\]

where

\[
G^\mu_{ij}(\mathbf{p}', \mathbf{p}; t_2, t_1) = \sum_{x_1, x_2} e^{-i \mathbf{p}' \cdot x_2} e^{i (\mathbf{p}' - \mathbf{p}) \cdot x_1} \langle \Omega | \chi_i(x_2) j^\mu(x_1) \chi_j(0) | \Omega \rangle
\]

is the matrix constructed from the three-point correlation functions of the original operators \(\{ \chi_i \}\).
To eliminate the time dependence of the three-point correlation function, we construct the ratio

\[
R_{\alpha}^{\mu}(p', p; t_2, t_1) = \left( \frac{G_{\alpha}^{\mu}(p', p; t_2, t_1) G_{\alpha}^{\mu}(p, p'; t_2, t_1)}{G_{\alpha}(p'; t_2) G_{\alpha}(p; t_2)} \right)^{1/2}
\]
Extracting Form Factors from Lattice QCD

- To eliminate the time dependence of the three-point correlation function, we construct the ratio

\[ R_\alpha^\mu(p', p; t_2, t_1) = \left( \frac{G_\alpha^\mu(p', p; t_2, t_1)}{G_\alpha(p; t_2)} \frac{G_\alpha^\mu(p, p'; t_2, t_1)}{G_\alpha(p'; t_2)} \right)^{1/2} \]

- To further simplify things, we define the reduced ratio

\[ \overline{R}_\alpha^\mu = \left( \frac{2E_\alpha(p)}{E_\alpha(p) + m_\alpha} \right)^{1/2} \left( \frac{2E_\alpha(p')}{E_\alpha(p') + m_\alpha} \right)^{1/2} R_\alpha^\mu \]
Current Matrix Element for Spin-1/2 Baryons

The current matrix element for spin-1/2 baryons has the form

\[ \langle p', s' | j^\mu | p, s \rangle = \left( \frac{m_\alpha^2}{E_\alpha(p)E_\alpha(p')} \right)^{1/2} \times \]
\[ \times \bar{u}(p') \left( F_1(q^2) \gamma^\mu + i F_2(q^2) \sigma^{\mu\nu} \frac{q^\nu}{2m_\alpha} \right) u(p) \]
Current Matrix Element for Spin-1/2 Baryons

The current matrix element for spin-1/2 baryons has the form

\[ \langle p', s'| j^\mu | p, s \rangle = \left( \frac{m^2_\alpha}{E_\alpha(p)E_\alpha(p')} \right)^{1/2} \times \]

\[ \times \bar{u}(p') \left( F_1(q^2) \gamma^\mu + i F_2(q^2) \sigma^{\mu\nu} \frac{q^\nu}{2m_\alpha} \right) u(p) \]

- The Dirac and Pauli form factors are related to the Sachs form factors through

\[ G_E(q^2) = F_1(q^2) - \frac{q^2}{(2m_\alpha)^2} F_2(q^2) \]

\[ G_M(q^2) = F_1(q^2) + F_2(q^2) \]
Sachs Form Factors for Spin-1/2 Baryons

- A suitable choice of momentum \( q = (q, 0, 0) \) and the (implicit) Dirac matrices allows us to directly access the Sachs form factors:
  - for \( G_E \): using \( \Gamma^\pm_4 \) for both two- and three-point,
    \[
    G^\alpha_E(q^2) = \mathcal{R}^4_\alpha(q, 0; t_2, t_1)
    \]
  - for \( G_M \): using \( \Gamma^\pm_4 \) for two-point and \( \Gamma^\pm_j \) for three-point,
    \[
    \epsilon_{ijk} q^i G^\alpha_M(q^2) = (E_\alpha(q) + m_\alpha) \mathcal{R}^k_\alpha(q, 0; t_2, t_1)
    \]
  - where for positive parity states,
    \[
    \Gamma_{\pm j}^+ = \frac{1}{2} \begin{bmatrix}
      \sigma_j & 0 \\
      0 & 0
    \end{bmatrix}
    \quad \Gamma_4^+ = \frac{1}{2} \begin{bmatrix}
      I & 0 \\
      0 & 0
    \end{bmatrix}
    \]
    and for negative parity states,
    \[
    \Gamma_{\pm j}^- = -\gamma_5 \Gamma_{\pm j}^+ \gamma_5 = -\frac{1}{2} \begin{bmatrix}
      0 & 0 \\
      0 & \sigma_j
    \end{bmatrix}
    \quad \Gamma_4^- = -\gamma_5 \Gamma_4^+ \gamma_5 = -\frac{1}{2} \begin{bmatrix}
      0 & 0 \\
      0 & I
    \end{bmatrix}
    \]
Negative Parity Nucleon: Five-quark Operators: cssm

\[ n_s = 35 + 200 \]

\[ 1 \rightarrow \chi_1 + \chi_2 \\
2 \rightarrow \chi_1 + \chi_2 + \chi_5 \\
3 \rightarrow \chi_1 + \chi_2 + \chi_5' \\
4 \rightarrow \chi_1 + \chi_2 + \chi_5 + \chi_5' \\
5 \rightarrow \chi_1 + \chi_5 + \chi_5' \\
6 \rightarrow \chi_2 + \chi_5 + \chi_5' \\
7 \rightarrow \chi_5 + \chi_5' \]
Negative Parity Nucleon Scattering Thresholds

• “Searching for low-lying multi-particle thresholds in lattice spectroscopy,”
M. S. Mahbub, et al. [CSSM],
Annals Phys. 342, 270 (2014)
arXiv:1310.6803 [hep-lat]

• “Lattice baryon spectroscopy with multi-particle interpolators,”
Adrian Kiratidis, Waseem Kamleh, Derek Leinweber, Benjamin Owen
[CSSM]
arXiv:1501.07667 [hep-lat].
Negative Parity Nucleon Spectrum: Lang and Verduci

![Graph showing the spectrum of negative parity nucleons with energy (E) on the y-axis and $t$ on the x-axis, with data points and error bars indicating experimental results and theoretical predictions for $N_-$ and $N_-, N\pi$.](image_url)
Negative Parity Nucleon Spectrum: Lang and Verduci

- Small correlation matrix: $\chi_1 + \chi_2 \times 2$ smearings $= 4 \times 4$
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• Did not construct projected correlators.
• Limited Euclidean time evolution.
Negative Parity Nucleon Spectrum: Lang and Verduci

- Small correlation matrix: $\chi_1 + \chi_2 \times 2$ smearings = $4 \times 4$
- Did not construct projected correlators.
- Limited Euclidean time evolution.
- Adding $N\pi$ sufficient but not necessary. *cf.* CSSM & JLab results

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Many groups (BGR, Cyprus, $\chi$QCD, CSSM) consider the following local interpolating fields

$$\chi_1(x) = \epsilon^{abc} (u^T a(x) \gamma_5 d^b(x)) u^c(x),$$

$$\chi_2(x) = \epsilon^{abc} (u^T a(x) C d^b(x)) \gamma_5 u^c(x).$$
Hybrid Baryons: Hadron Spectrum Collaboration

\[ m / \text{GeV} \]

\[ \begin{array}{cccc}
\frac{1}{2}^+ & \frac{3}{2}^+ & \frac{5}{2}^+ & \frac{7}{2}^+ \\
\end{array} \]

\[ N^* \]

\[ m_\pi = 396 \text{ MeV} \]